Improving the Fidelity of Quantum Cloning by Fast Cycling away the Unwanted Transition

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The fidelity of quantum cloning is very often limited by the accompanying unwanted transitions. We show how the fidelity can be improved by using a coherent field to cycle away the unwanted transitions. We demonstrate this explicitly in the context of the model of Simon *et al.* [Phys. Rev. Lett. **84**, 2993 (2000)]. We also investigate the effects of the number of atoms and initial atomic coherence on the quality of quantum cloning.

I. INTRODUCTION

Quantum no-cloning theorem states that it is impossible to clone perfectly an arbitrary *unknown* pure quantum state [1], or a mixed quantum state [2]. The origin of this theorem can be traced to the linearity of quantum mechanics. However imperfect quantum cloning is possible. An optimal $1 \rightarrow 2$ imperfect cloner has been proposed [3], which is universal for all input qubits and is compatible with no-signalling constraints too [4]. It has been generalized [5,6] for the case of $N \rightarrow M$ $(M \geq N)$ cloner. The upper bound for the fidelity of a $N \rightarrow M$ cloner has been established [7]. Results are also available for the optimal cloning of arbitrary pure and mixed states in d-dimensions $(d \geq 2)$ [6,8]. Besides, cloning of entangled states [9] and of Gaussian-distributed quantum variables [10] has been considered. Quantum cloning was originally discussed in terms of polarization states of photons [1,11,12]. Making use of two two-level atoms with orthogonal transition dipole moments, Mandel [12] proved that one can make the output of the photon amplifier independent of the input polarization. Perfect cloning of photonic states has been proved impossible due to inevitable coexistence of spontaneous emission with stimulated emission process [11,12]. Simon and co-workers [13] have proposed a quantum cloning machine (QCM) consisting of three-level atoms. They have shown that the quantum cloning of a single input qubit (polarization state of a photon) using a V-system is possible with a fidelity 5/6 at least for shorter interaction times. This value is optimal for a $1 \rightarrow 2$ quantum cloner [3]. Similar results for Λ-systems have been reported [14]. Finally note that the state-dependent cloning has also been studied extensively [15]. In this paper, we address the question if the fidelity of a V-system cloner can be improved by using some type of external field.

The organization of this paper is as follows. In Sec. II, we briefly outline the cloning scheme introduced by Simon *et al.*. We prove the universality of their scheme by using a new basis for the states of the radiation fields. We study the effects of atomic coherence on cloning as well. In Sec. III, we examine the reason for imperfect fidelity in a V-scheme and introduce a way to improve this fidelity by using a coherent field. The external field cycles away the unwanted transition responsible for spontaneous emission. We present both analytical and numerical results for the fidelity. In Sec. IV, we show how further improvement in fidelity can be obtained by considering a cloner consisting of two V-systems.

II. QUANTUM CLONING BASED ON STIMULATED EMISSION IN A V-SYSTEM

A. Optimal Photon Cloner with a V-configuration

In a recent paper, Simon *et al.* [13] have proposed a new scheme for quantum cloning of a photonic qubit. They considered a cloning device consisting of an ensemble of atoms trapped inside a cavity. The relevant atomic transitions correspond to the V-system. These are three-level systems with two degenerate excited states $|e_1\rangle$ and $|e_2\rangle$ and a common ground level $|g\rangle$. The ground level is coupled to the excited states by two orthogonal field modes a_1 and a_2, respectively.

In the interaction picture [16], the effective Hamiltonian under dipole and rotating wave approximations [17,18] can be written as

$$H_I = \hbar g \sum_{k=1}^{N} (\sigma_{+1}^k a_1 + \sigma_{+2}^k a_2) + \text{ H. c.}, \tag{1}$$

where g is the coupling constant between the field-modes and the atoms, $\sigma_{+1(2)}^k$'s $[= (|e_{1(2)}\rangle\langle g|)_k]$ are the raising operators between the corresponding states of the k-th atom. Here g is assumed to be equal for all the atoms [19]. Also both the cavity-modes are assumed to be resonant with the corresponding atomic transitions.

Let each atom be prepared initially in a mixed excited state

$$\rho = \frac{1}{2}(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|) \tag{2}$$

and the photonic qubit be prepared in a state $a_1^\dagger|0,0\rangle \equiv |1,0\rangle$. The time-development operator $U = e^{-iH_I t}$ will provide the time-evolution of the entire (atom+photon) system and this is used to study quantum cloning.

The quality of cloning is quantified by a parameter called fidelity [20], defined as

$$F_{\text{clone}} = \sum_{k+l\,\geq\,2} p'(k,l)\left(\frac{k}{k+l}\right), \tag{3a}$$

where

$$p'(k,l) = \frac{p(k,l)}{1 - p(1,0) - p(0,1)}. \tag{3b}$$

Here $p(k,l)$ represents the probability of finding k photons in the initial mode and l photons in the orthogonal mode a_2 in the evolved state. It should be noted that for an ensemble of N atoms, the maximum value of k will be $N+1$, which corresponds to all the atoms decaying to the ground state through the emission of the a_1-photon. Thus F_{clone} is a kind of an average of the relative frequency of photons in initial mode a_1 in the final state. The final time-evolved entangled state contains the field components with various combination of photon numbers in two modes.

As shown by Simon *et al.*, the fidelity is optimal for short interaction times and for $N = 6$. It decreases for later times. They have explained this behavior in terms of stimulated and spontaneous emissions on the transitions $|e_1\rangle \to |g\rangle$ and $|e_2\rangle \to |g\rangle$, respectively. If there is an extra photon in a_1-mode, it can be considered as a clone of the initial qubit. Note that the probability to get a clone is reduced if there is an extra photon in the other (a_2) mode, which is due to spontaneous emission. Thus the fidelity is reduced to a value equal to or less than 5/6.

B. Question of Universality of Cloning by a V-system

It is mentioned in Ref. [13] that the above scheme is universal, i.e., the V-system cloner can clone even any arbitrary photonic qubit, say, $(\alpha a_1^\dagger + \beta a_2^\dagger)|0,0\rangle$ with the same non-unity fidelity. Simon *et al.* argued that this is because the initial mixed state and the Hamiltonian are invariant under a unitary transformation. In what follows, we demonstrate explicitly this universality by changing the basis to a general qubit state.

Consider a single atom in V-configuration, initially prepared in a superposition of the two excited states

$$|s\rangle = \frac{1}{\sqrt{2}}\left(|e_1\rangle + e^{i\theta}|e_2\rangle\right). \tag{4}$$

Let the photon be in a superposition state

$$b_1^\dagger|0,0\rangle \equiv \left(\alpha a_1^\dagger + \beta a_2^\dagger\right)|0,0\rangle \equiv \alpha|1,0\rangle + \beta|0,1\rangle \tag{5}$$

as well, where α and β are the complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. Note that an average over θ will give an initial state of the atom, which is a mixed state. We consider the photon as a qubit [21], which can be in any linear superposition of the two orthogonal states. Let us define the basis state $b_2^\dagger|0,0\rangle$, which is orthogonal to (5). The new operators b_1 and b_2 must satisfy the commutation relations

$$[b_1, b_2] = [b_1, b_2^\dagger] = 0. \tag{6}$$

Using Eqs. (5) and (6), we get

$$b_2^\dagger \equiv \beta^* a_1^\dagger - \alpha^* a_2^\dagger. \tag{7}$$

The time-evolution of the entire system is determined by the Schrödinger equation

$$i\hbar \frac{d}{dt}|\Psi(t)\rangle = H_I |\Psi(t)\rangle, \tag{8}$$

where H_I is given by Eq. (1) for $N = 1$. We expand $|\Psi(t)\rangle$ in terms of the relevant basis states. Starting with the initial conditions [Eqs. (4) and (5)], these relevant states were found to be

$$|e_1\rangle|1,0\rangle \ ; \ |e_2\rangle|1,0\rangle \ ; \ |g\rangle|2,0\rangle \ ; \ |g\rangle|1,1\rangle \ ; \ |g\rangle|0,2\rangle \ ; \ |e_1\rangle|0,1\rangle \ ; \ |e_2\rangle|0,1\rangle. \tag{9}$$

The only non-zero expansion amplitudes C_α^{mn} at time $t = 0$ are

$$C_{e_1}^{10}(0) = \frac{\alpha}{\sqrt{2}} \ ; \ C_{e_1}^{01}(0) = \frac{\beta}{\sqrt{2}} \ ; \ C_{e_2}^{10}(0) = \frac{\alpha}{\sqrt{2}} e^{i\theta} \ ; \ C_{e_2}^{01}(0) = \frac{\beta}{\sqrt{2}} e^{i\theta}, \tag{10}$$

where the subscript (superscript) denotes the atom (photons) in the state α (m, n). Then all the expansion amplitudes can be evaluated in closed form with the following results :

$$\begin{aligned}
C_{e_1}^{10}(t) &= \frac{\alpha}{\sqrt{2}} \cos\left(\sqrt{2}gt\right), \\
C_g^{20}(t) &= -i\frac{\alpha}{\sqrt{2}} \sin\left(\sqrt{2}gt\right), \\
C_{e_2}^{10}(t) &= \frac{1}{2\sqrt{2}} \left[\left(\alpha e^{i\theta} - \beta\right) + \left(\beta + \alpha e^{i\theta}\right) \cos\left(\sqrt{2}gt\right)\right], \\
C_{e_1}^{01}(t) &= \frac{1}{2\sqrt{2}} \left[\left(\beta - \alpha e^{i\theta}\right) + \left(\beta + \alpha e^{i\theta}\right) \cos\left(\sqrt{2}gt\right)\right], \\
C_g^{11}(t) &= -\frac{i}{2} \left(\beta + \alpha e^{i\theta}\right) \sin\left(\sqrt{2}gt\right), \\
C_g^{02}(t) &= -i\frac{\beta}{\sqrt{2}} e^{i\theta} \sin\left(\sqrt{2}gt\right), \\
C_{e_2}^{01}(t) &= \frac{\beta}{\sqrt{2}} e^{i\theta} \cos\left(\sqrt{2}gt\right).
\end{aligned} \tag{11}$$

The reduced density matrix of the field is defined by

$$\rho_F = \mathrm{Tr}_A(|\Psi(t)\rangle\langle\Psi(t)|). \tag{12}$$

Using Eq. (12), the probability $\tilde{p}(k, l)$ that k photons will be in b_1-mode and l photons in b_2-mode can be written in terms of b-operators as

$$\tilde{p}(k, l) = \langle 0, 0| \frac{b_1^k \, b_2^l}{\sqrt{k! \, l!}} \, \rho_F \, \frac{b_1^{\dagger k} \, b_2^{\dagger l}}{\sqrt{k! \, l!}} |0, 0\rangle. \tag{13}$$

Further in order to get the initial atomic state used by Simon *et al.*, we average $\tilde{p}(k, l)$ over all values of θ. A lengthy derivation yields the following :

$$\tilde{p}(2, 0) = \frac{1}{2} \sin^2\left(\sqrt{2}gt\right), \tag{14a}$$

$$\tilde{p}(1, 1) = \frac{1}{4} \sin^2\left(\sqrt{2}gt\right), \tag{14b}$$

$$\tilde{p}(0, 1) = \frac{1}{8} \cos^2\left(\sqrt{2}gt\right) - \frac{1}{4} \cos\left(\sqrt{2}gt\right) + \frac{1}{8}, \tag{14c}$$

$$\tilde{p}(1, 0) = \frac{5}{8} \cos^2\left(\sqrt{2}gt\right) + \frac{1}{4} \cos\left(\sqrt{2}gt\right) + \frac{1}{8}. \tag{14d}$$

The Eqs. (14) lead to the expression for the fidelity as

$$F_{\text{clone}} = \frac{\tilde{p}(2, 0) + \frac{1}{2}\tilde{p}(1, 1)}{\tilde{p}(2, 0) + \tilde{p}(1, 1)} = \frac{5}{6}. \tag{15}$$

Clearly the fidelity does not depend upon α and β. This reflects the fact that the V-scheme is universal as a cloner, which can clone even a general superposition of two orthogonal modes of the field, albeit imperfectly. A similar result has been reported recently using a different method [14].

C. Effect of Atomic Coherence on Quantum Cloning

We next examine the effect of initial atomic coherence on the fidelity of cloning. The probabilities $\tilde{p}(k,l)$ in Eq. (13) can be written as

$$\tilde{p}(2,0) = \left(\frac{1}{2} + \alpha\beta\cos\theta\right)\sin^2\left(\sqrt{2}gt\right), \tag{16a}$$

$$\tilde{p}(1,1) = \left(\frac{1}{4} - \frac{1}{2}\alpha\beta\cos\theta\right)\sin^2\left(\sqrt{2}gt\right), \tag{16b}$$

$$\tilde{p}(0,1) = \frac{1}{8}\cos^2\left(\sqrt{2}gt\right) - \frac{1}{4}\cos\left(\sqrt{2}gt\right) + \frac{1}{8} - \frac{1}{4}\alpha\beta\cos\theta\sin^2\left(\sqrt{2}gt\right) + \frac{1}{2}\alpha\beta\cos\theta\left\{1-\cos\left(\sqrt{2}gt\right)\right\}\cos\left(\sqrt{2}gt\right), \tag{16c}$$

$$\tilde{p}(1,0) = \frac{5}{8}\cos^2\left(\sqrt{2}gt\right) + \frac{1}{4}\cos\left(\sqrt{2}gt\right) + \frac{1}{8} - \frac{1}{4}\alpha\beta\cos\theta\sin^2\left(\sqrt{2}gt\right) - \frac{1}{2}\alpha\beta\cos\theta\left\{1-\cos\left(\sqrt{2}gt\right)\right\}\cos\left(\sqrt{2}gt\right). \tag{16d}$$

Using Eqs. (3), the fidelity of cloning by a single V-atom can be written as

$$F_{\text{clone}} = \frac{5 + 6\alpha\beta\cos\theta}{6 + 4\alpha\beta\cos\theta}, \tag{17}$$

which has two free parameters α and θ. They can be chosen such that the fidelity becomes unity. This happens (see Fig. 1) only when

$$\alpha = \beta = \frac{1}{\sqrt{2}} \quad ; \quad \theta = 0, 2\pi. \tag{18}$$

In other words, an equal weight superposition state of the excited V-atom can produce true clone of an initial qubit in $(a_1 + a_2)/\sqrt{2}$-mode. This signifies that even a V-system can be used as a perfect cloner for certain initial states. The Fig. 1 also shows regions, where $F_{\text{clone}} > 5/6$.

It is also clear from Eq. (17) that for $\theta = \pi/2$ and $3\pi/2$, the fidelity of cloning by a V-atom becomes 5/6 for all values of α. This means that the universality and optimality can be achieved even for an initial superposition of atomic states. Thus we infer that the initial mixed state of the atoms is *not* an essential criterion for achieving the best fidelity of universal cloning and that coherence between the atomic excited states does influence the fidelity of cloning.

III. A METHOD TO IMPROVE THE FIDELITY OF THE V-SCHEME

The fidelity of the V-system as a cloner is 5/6, which has been proved to be optimal [14]. Here we outline a proposal to improve this value. A fidelity close to unity can be achieved for certain values of the interaction times and for initial states of the atom, which are in coherent superposition of the two excited states. Using Eqs. (3), the fidelity for a single-atom cloner can be written as

$$F_{\text{clone}} = 1 - \frac{1}{2\left(1 + \frac{p(2,0)}{p(1,1)}\right)}. \tag{19}$$

This shows that in the limit $p(1,1) \to 0$, $F_{\text{clone}} \to 1$. In other words, a non-zero value of $p(1,1)$ takes F_{clone} away from unity. Note that $p(2,0)$ [$p(1,1)$] is the probability of the stimulated [spontaneous] emission in a_1[a_2]-mode. Thus to improve the fidelity one should reduce the probability $p(1,1)$ of this spontaneous emission, which is caused by the atomic population in $|e_2\rangle$-level. One possible way of doing this is to *cycle* this population away, so that this unwanted spontaneous decay does not occur very often. We show that it can be done by applying a classical pump field, which causes the population in the state $|e_2\rangle$ to pulsate between a metastable state $|f\rangle$ and $|e_2\rangle$.

Consider a four-level atomic configuration as shown in Fig. 2. The excited states $|e_1\rangle$ and $|e_2\rangle$ are coupled to the common ground level $|g\rangle$ through the two orthogonal modes a_1 and a_2 of the quantized electromagnetic field, respectively. The coupling constant between each of these excited states and $|g\rangle$ is g. A classical field couples the state $|e_2\rangle$ with the metastable state $|f\rangle$. The corresponding Rabi frequency is $2gG$, where G is a multiplying factor and is related to the number of photons in the classical field. We assume all the fields to be resonant with the corresponding atomic transitions.

We start with a single atom prepared in a superposition of the excited states [vide Eq. (4)]. The initial photonic qubit is in a_1-mode. We will work in the interaction picture. Then using the rotating wave approximation to eliminate the fast-oscillating energy non-conserving terms, we obtain the effective Hamiltonian as

$$H_I = \hbar g\left[\sigma_{+1}a_1 + \sigma_{+2}a_2 + G|e_2\rangle\langle f|\right] + \text{H.c.}, \tag{20}$$

where $\sigma_{+1(2)} = |e_{+1(2)}\rangle\langle g|$ are the raising operators of the atom as defined in the previous section. The time-evolution of the entire system can be determined by the Schrödinger equation [see Eq. (8)]. Now starting with the initial basis states [Eqs. (4) and (5) for $\alpha = 1$], we find the possible relevant basis vectors as

$$|e_1\rangle|1,0\rangle\ ;\ |e_2\rangle|1,0\rangle\ ;\ |f\rangle|1,0\rangle\ ;\ |g\rangle|1,1\rangle\ ;\ |g\rangle|2,0\rangle\ ;\ |e_1\rangle|0,1\rangle. \tag{21}$$

We expand $|\Psi(t)\rangle$ in terms of these basis states. The initial conditions can be written in terms of the expansion amplitudes as

$$C^{10}_{e_1}(0) = \frac{1}{\sqrt{2}} \quad ; \quad C^{10}_{e_2}(0) = \frac{1}{\sqrt{2}}\, e^{i\theta}. \tag{22}$$

Using Eqs. (8) and (20), we obtain the following first order differential equations for the expansion amplitudes :

$$\dot{C}^{10}_{e_1}(t) = -\sqrt{2}igC^{20}_{g}(t), \tag{23a}$$
$$\dot{C}^{20}_{g}(t) = -\sqrt{2}igC^{10}_{e_1}(t), \tag{23b}$$
$$\dot{C}^{10}_{e_2}(t) = -igC^{11}_{g}(t) - igGC^{10}_{f}(t), \tag{23c}$$
$$\dot{C}^{10}_{f}(t) = -igGC^{10}_{e_2}(t), \tag{23d}$$
$$\dot{C}^{11}_{g}(t) = -igC^{01}_{e_1}(t) - igC^{10}_{e_2}(t), \tag{23e}$$
$$\dot{C}^{01}_{e_1}(t) = -igC^{11}_{g}(t). \tag{23f}$$

From the conditions (22), we find the following solutions for the Schrödinger amplitudes :

$$C^{11}_{g}(t) = A\sin\left(\frac{\Omega_1}{\sqrt{2}}gt\right) + B\sin\left(\frac{\Omega_2}{\sqrt{2}}gt\right), \tag{24a}$$
$$C^{10}_{f}(t) = \frac{1}{2G}\left[(\Omega_1^2-4)A\sin\left(\frac{\Omega_1}{\sqrt{2}}gt\right) + (\Omega_2^2-4)B\sin\left(\frac{\Omega_2}{\sqrt{2}}gt\right)\right], \tag{24b}$$
$$C^{10}_{e_2}(t) = \frac{i}{2\sqrt{2}G^2}\left[\Omega_1(\Omega_1^2-4)A\cos\left(\frac{\Omega_1}{\sqrt{2}}gt\right) + \Omega_2(\Omega_2^2-4)B\cos\left(\frac{\Omega_2}{\sqrt{2}}gt\right)\right], \tag{24c}$$
$$C^{01}_{e_1}(t) = -\frac{i}{2\sqrt{2}G^2}\left[\Omega_1(\Omega_1^2-2G^2-4)A\cos\left(\frac{\Omega_1}{\sqrt{2}}gt\right) + \Omega_2(\Omega_2^2-2G^2-4)B\cos\left(\frac{\Omega_2}{\sqrt{2}}gt\right)\right], \tag{24d}$$
$$C^{10}_{e_1}(t) = \frac{1}{\sqrt{2}}\cos\left(\sqrt{2}gt\right), \tag{24e}$$
$$C^{20}_{g}(t) = -\frac{i}{\sqrt{2}}\sin\left(\sqrt{2}gt\right), \tag{24f}$$

where

$$\Omega_1 = \left(G^2+2+\sqrt{G^4+4}\right)^{\frac{1}{2}} \quad , \quad \Omega_2 = \left(G^2+2-\sqrt{G^4+4}\right)^{\frac{1}{2}},$$

$$A = \frac{1}{2}ie^{i\theta}\frac{(\Omega_2^2-2G^2-4)}{\Omega_1\sqrt{G^4+4}},$$
$$B = -\frac{1}{2}ie^{i\theta}\frac{(\Omega_1^2-2G^2-4)}{\Omega_2\sqrt{G^4+4}}.$$

Using the relation (12), we get the reduced density matrix of the field. The diagonal element of this density matrix in field basis $|k,l\rangle$ gives the probability $p(k,l)$ that k photons are in a_1-mode and l photons are in a_2-mode. We have found that :

$$p(2,0) = |C_g^{20}(t)|^2 = \frac{1}{2}\sin^2\left(\sqrt{2}gt\right), \tag{25a}$$

$$p(1,1) = |C_g^{11}(t)|^2, \tag{25b}$$

$$p(1,0) = |C_f^{10}(t)|^2 + |C_{e_2}^{10}(t)|^2 + |C_{e_1}^{10}(t)|^2, \tag{25c}$$

$$p(0,1) = |C_{e_1}^{01}(t)|^2. \tag{25d}$$

The probability $p(1,1)$ after averaging over θ becomes

$$p_a(1,1) = \frac{1}{4(G^4+4)}\left[\frac{(\Omega_2^2-2G^2-4)^2}{\Omega_1^2}\sin^2\left(\frac{\Omega_1}{\sqrt{2}}gt\right) + \frac{(\Omega_1^2-2G^2-4)^2}{\Omega_2^2}\sin^2\left(\frac{\Omega_2}{\sqrt{2}}gt\right)\right]. \tag{26}$$

Hence the fidelity F_{clone} takes the following form :

$$F_{\text{clone}}(t) = \frac{p(2,0)+\frac{1}{2}p_a(1,1)}{p(2,0)+p_a(1,1)}. \tag{27}$$

We have plotted $F_{\text{clone}}(t)$ [Eq. (27)] as a function of time for $G=8$ in Fig. 3. It is found that as G increases, the fidelity becomes unity more often. Whenever $p_a(1,1)$ becomes zero, $F_{\text{clone}}(t)$ becomes unity. In fact, by introducing the classical pump field, we cycle the atomic population in the state $|e_2\rangle$ to the state $|f\rangle$ and back. This inhibits the spontaneous decay of the atom in the state $|e_2\rangle$ to the ground state $|g\rangle$. This cycling-recycling process goes on until $p(2,0)$ vanishes. There are two time-scales of oscillation of $F_{\text{clone}}(t)$. The faster small-amplitude oscillation is attributed to that of $p_a(1,1)$. This oscillation can be increased by G so that the atom effectively goes to the state $|f\rangle$ very frequently. This means that $F_{\text{clone}}(t)$ becomes close to unity more frequently. This happens until $p(2,0)$ goes to zero. The effect of spontaneous emission from the state $|f\rangle$ is ignored assuming that the time scale for this decay is much larger than that for Rabi oscillation between $|e_2\rangle$ and $|f\rangle$-levels.

IV. IMPROVEMENT OF FIDELITY OF CLONING WITH TWO ATOMS

In this section, we show that by considering two atoms one can improve the fidelity further for a larger domain of times if we adopt the use of the cycling field. We consider the case of two V-atoms. The interaction Hamiltonian is the same as in Eq. (1) with $N=2$. We assume that the entire system is in the initial state given by

$$|S\rangle = \frac{1}{2}\left(|e_1\rangle + e^{i\theta}|e_2\rangle\right)_{\text{A}}\left(|e_1\rangle + e^{i\theta}|e_2\rangle\right)_{\text{B}}|1,0\rangle. \tag{28}$$

The indices A and B refer to the two atoms. For this initial condition we need to use twenty basis states. Let $C_{\alpha\beta}^{mn}(t)$ represent the probability amplitude for finding $m(n)$ photons in $a_1(a_2)$-mode and the $A(B)$-atom in the state $\alpha(\beta)$ at time t. After a tedious calculation, we have found the following analytical results for these amplitudes :

$$C_{e_1\,g}^{20}(t) = C_{e_2\,g}^{20}(t) = -\frac{i}{\sqrt{20}}\sin\left(\sqrt{10}gt\right), \tag{29a}$$

$$C_{gg}^{30}(t) = -\frac{\sqrt{6}}{5}\sin^2\left(\frac{\sqrt{10}}{2}gt\right), \tag{29b}$$

$$C_{e_1\,e_1}^{10}(t) = \frac{3}{10} + \frac{1}{5}\cos\left(\sqrt{2}gt\right). \tag{29c}$$

In order to get all the remaining amplitudes, we solved the equations numerically using 5-th order Runge-Kutta-Fehlberg method. The averaged probabilities $p_a(k,l)$'s for the photons to exist in the (a_1,a_2)-basis have been evaluated from the reduced density matrix [Eq. (12)] of the field. We evaluate the fidelity $F_{\text{clone}}(t)$ using Eqs. (3). It is plotted in Fig. 3 as a function of time. It is found that $F_{\text{clone}}(t)$ decreases for later times starting from the optimal value of 5/6 as pointed out by Simon *et al.*. As clearly seen in Eq. (29b), there is a finite probability of obtaining three photons at the most in a_1-mode. This happens when both the atoms decay to the state $|g\rangle$ via a_1-channel.

We next calculate the fidelity in presence of the cycling field. We discussed in Sec. III how the applied field on the transition $|e_2\rangle \leftrightarrow |f\rangle$ can increase the fidelity. We now extend our considerations to two atoms. We use the level-configuration of Fig. 2. The interaction Hamiltonian of this system can be written as

$$H_I = \hbar g \sum_{k=1}^{2} \left[\sigma_{+1}^k a_1 + \sigma_{+2}^k a_2 + G|e_2\rangle_{k\ k}\langle f|\right] + \text{H. c.}. \tag{30}$$

We assume that each of the atoms is initially prepared in a superposition of $|e_1\rangle$ and $|e_2\rangle$ states [see (4)] and a single photon is in a_1-mode. Then $|\Psi(t)\rangle$ can be expanded in terms of thirty relevant basis states. These states span the entire Hilbert space of the system consisting of two atoms and the two field modes. The time-dependence of the corresponding expansion amplitudes can be evaluated analytically by Laplace transform. We present the numerical results as the analytical results are too complex. The results for the fidelity $F_{\rm clone}(t)$ are shown in Fig. 4. It is clear from this figure that the cycling field makes the fidelity close to unity for almost all times. Note that the fidelity of the two-atom cloner can be expressed as

$$F_{\rm clone}(t) = 1 - \frac{\frac{1}{3}p_a(2,1) + \frac{2}{3}p_a(1,2) + \frac{1}{2}p_a(1,1) + p_a(0,2)}{p_a(2,0) + p_a(3,0) + p_a(2,1) + p_a(1,2) + p_a(1,1) + p_a(0,2)}. \tag{31}$$

Obviously, it becomes unity only if the probabilities of spontaneous emission in a_2-mode (both in presence and in absence of photons in a_1-mode) are zero. Then all the photons present in the cavity would be in a_1-mode. However due to complex nature of time-dependence of $p_a(k,l)$'s, one does not find any periodicity in variation of $F_{\rm clone}(t)$. On comparing Figs. 3 and 4, we find that the fidelity of cloning is higher for the case of a cloner consisting of two atoms than for the case of a cloner consisting of a single atom.

V. CONCLUSIONS

In conclusion, we have proposed how the fidelity of a V-system-based quantum cloner can be improved by inhibiting the spontaneous emission effects on the unwanted transition. We showed that this can be done by applying a classical coherent field, which cycles the atom in the state $|e_2\rangle$ to some other metastable state. The fidelity remains close to unity over large intervals of time. The fidelity of V-scheme improves further by using two atoms inside the cavity provided we continue to use a cycling field. However we must add that by using the cycling field we, as expected, loose the universality of the scheme. We also examined the fidelity of cloning if the initial atomic state is a coherent superposition of the two excited states. There is a parameter domain in which the fidelity could be larger.

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FIG. 1. The variation of the fidelity of cloning by a single V-system with θ is plotted here. It is seen that only for $\theta = \pi/2$ and $3\pi/2$, the fidelity becomes optimal.

FIG. 2. An atomic level configuration to improve the fidelity of quantum cloning. Here the classical field with Rabi frequency $2gG$ cycles the atoms through the metastable state $|f\rangle$. The atom is inside a cavity, which allows only two field-modes a_1 and a_2.

FIG. 3. This figure shows the time-dependence of the fidelity of a four-level atomic cloner comprising a single atom for the external field parameter $G = 8$.

FIG. 4. The fidelity of a two-atom cloner is plotted as a function of time for no cycling field ($G = 0$, dashed curve) and in presence of the cycling field ($G = 8$, solid curve). It is obvious that application of a classical field improves the fidelity.

[1] W. K. Wootters and W. H. Zurek, Nature (London) **299**, 802 (1982).

[2] H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa, and B. Schumacher, Phys. Rev. Lett. **76**, 2818 (1996).
[3] V. Bužek and M. Hillery, Phys. Rev. A **54**, 1844 (1996).
[4] N. Gisin, Phys. Lett. A **242**, 1 (1998).
[5] N. Gisin and S. Massar, Phys. Rev. Lett. **79**, 2153 (1997).
[6] R. F. Werner, Phys. Rev. A **58**, 1827 (1998).
[7] D. Bruß, A. Ekert, and C. Macchiavello, Phys. Rev. Lett. **81**, 2598 (1998).
[8] V. Bužek and M. Hillery, Phys. Rev. Lett. **81**, 5003 (1998) (For a review, see V. Bužek and M. Hillery, e-print : **quant-ph**/9801009) ; M. Keyl and R. F. Werner, J. Math. Phys. **40**, 3283 (1999) ; N. J. Cerf, J. Mod. Opt. **47**, 187 (2000).
[9] M. Koashi and N. Imoto, Phys. Rev. Lett. **81**, 4264 (1998).
[10] N. J. Cerf, A. Ipe, and X. Rottenberg, Phys. Rev. Lett. **85**, 1754 (2000) ; N. J. Cerf and S. Iblisdir, Phys. Rev. A **62**, 040301(R) (2000).
[11] P. W. Milonni and M. L. Hardies, Phys. Lett. **92A**, 321 (1982).
[12] L. Mandel, Nature **304**, 188 (1983).
[13] C. Simon, G. Weihs, and A. Zeilinger, Phys. Rev. Lett. **84**, 2993 (2000).
[14] J. Kempe, C. Simon, and G. Weihs, Phys. Rev. A **62**, 032302 (2000).
[15] M. Hillery and V. Bužek, Phys. Rev. A **56**, 1212 (1997) ; D. Bruß, D. P. DiVincenzo, A. Ekert, C. A. Fuchs, C. Macchiavello, and J. A. Smolin, *ibid.* **57**, 2368 (1998).
[16] J. J. Sakurai, *Modern Quantum Mechanics*, (Addison Wesley, New York, 1994), p. 316.
[17] L. Allen and J. H. Eberly, *Optical Resonance and Two-level Atoms*, (John Wiley and Sons, New York, 1975), p. 41.
[18] M. O. Scully and S. Zubairy, *Quantum Optics*, (Cambridge University Press, Cambridge, UK, 1997), p. 196.
[19] This means that all the atoms are in equivalent positions relative to the cavity mode. This condition can be achieved by trapping the atoms inside the cavity, so that the field interacting with the atom is the same for all atoms. This kind of trapping of atoms has been reported recently by J. Ye, D. W. Vernooy, and H. J. Kimble, **quant-ph**/9908007.
[20] The concept of fidelity for information transfer was first introduced by Shannon [See C. E. Shannon and W. Weaver, *The Mathematical Theory of Communication*, (University of Illinois Press, Urbana, IL, 1963), p. 108.] in classical sense and was reformulated later in contexts of quantum coding [21] and cloning [3].
[21] B. Schumacher, Phys. Rev. A **51**, 2738 (1995).